Study of rarefied shear flow by the discrete velocity method

By JAMES E. BROADWELL

TRW Space Technology Laboratories, Redondo Beach, California

(Received 30 December 1963)

The application of a simple discrete velocity model to low Mach number Couette and Rayleigh flow is investigated. In the model, the molecular velocities are restricted to a finite set and in this study only eight equal speed velocities are allowed. The Boltzmann equation is reduced by this approximation to a set of coupled differential equations which can be solved in closed form. The fluid velocity and shear stress in Couette flow are in approximate accord with those of Wang Chang & Uhlenbeck (1954) and of Lees (1959) over the complete range of Knudsen number. Similarly, the Rayleigh flow solution is remarkably like those found by other investigators using moment methods.

1. Introduction

In elementary kinetic theory the molecules are assumed to move in only three (mutually perpendicular) directions and at constant speed. It is well known that the expressions for the equation of state and for the transport properties that come from this simple model are surprisingly similar to those derived from the most elaborate analyses. This outcome suggests that the same idea be applied in the study of rarefied gas motion, i.e. that the molecules be restricted to a finite set of prescribed velocities with the fluid motion being achieved, of course, by non-uniform distribution of the molecules among the allowed velocities.

This approach is also suggested by a common approximation in radiative transport problems, namely the assumption that radiation travels only along discrete rays, an approximation which reduces the governing integro-differential equation to a coupled set of differential equations (Chandrasekhar 1960). The application of such a procedure to the Boltzmann equation has been mentioned (Krook 1955; Gross 1960) but seems not to have been pursued.

This paper describes the low Mach number Couette and Rayleigh flow of a gas in which the molecules move in only eight directions and at constant speed. As is implied above the aim will be to obtain solutions of the Boltzmann equation for this gas and thus to describe these flows at all Knudsen numbers.

2. The discrete velocity model

The Boltzmann equation may be written in the form

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial t}\right)_c = G-L, \tag{1}$$

in which f is the distribution function depending on the time and space variables t, x, y, z and on the molecular velocity $\mathbf{v}$ with components u, v, w. The rate of change of f due to collisions, $(\partial f/\partial t)_c$, is written as the gain minus the loss, $G-L$. Now if the molecules occupy, between collisions, a finite set of cells located at $\mathbf{v}_i$ in velocity space, equation (1) can be written

$$\frac{\partial N_i}{\partial t}+u_i\frac{\partial N_i}{\partial x}+v_i\frac{\partial N_i}{\partial y}+w_i\frac{\partial N_i}{\partial z}=\left(\frac{\partial N_i}{\partial t}\right)_c=G_i-L_i, \tag{2}$$

in which N_i is the number of molecules per unit volume with velocity $\mathbf{v}_i$.

Before evaluating (G_i-L_i) for eight cells, consider as a simpler example the two-dimensional gas lying in the (u,v)-plane with the four velocities, of magnitude $\bar{c}$, sketched in figure 1(a). The rate of change of N_1, for instance, due to collisions is

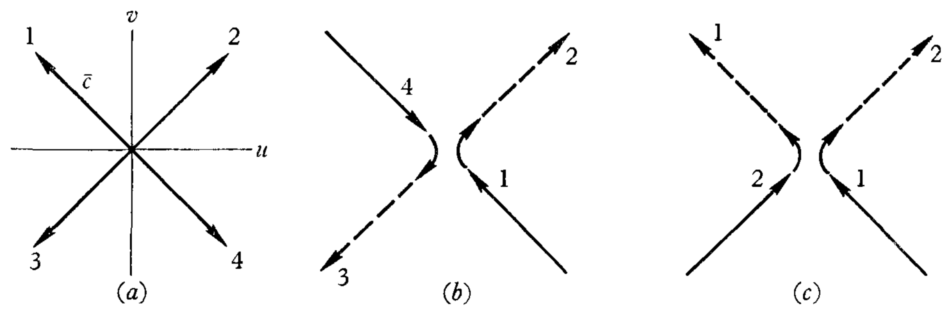

FIGURE 1. Collisions in a two-dimensional gas.

determined as follows. Loss of molecules from cell 1 occurs only when these molecules collide with those in cell 4, for collision with occupants of cells 2 or 3 results only in an exchange of cells (see figures 1(b) and 1(c)). Thus the loss from cell 1 can be written

$$L_1 = v_r S_e N_1 N_4 = 2\bar{c}S_e N_1 N_4,$$

where v_r is the relative speed, equal to $2\bar{c}$, and S_e is the effective collision cross-section, i.e. the cross-section which deflects collision partners from cells 1 and 4 to 2 and 3. These same arguments show, of course, that molecules are thrown into cell 1 from collisions 2–3; therefore

$$G_1 = 2\bar{c}S_e N_2 N_3$$

and
$$\left(\frac{\partial N_1}{\partial t}\right)_c = 2\bar{c}S_e(N_2N_3-N_1N_4).$$

The expressions for the other cells can be written in the same way.

Now consider eight velocity cells symmetrically placed in the eight quadrants as shown in figure 2. The magnitude of the velocity is again $\bar{c}$ and $q=\bar{c}/3^{\frac{1}{2}}$ is the velocity component along the co-ordinate axes. With this arrangement (G_i-L_i) in equation (2) can be determined as follows.

Begin with cell 1. To find the rate of change of N_1 due to collisions we must first determine which of its possible collisions cause a net loss or gain from cell 1 as distinguished from those in which the collision partners simply exchange cells.

Refer to table 1 in which the signs of the velocity components are listed. Molecules may jump only between pairs of cells in which there is the same combination of signs for u, v, and w. Thus the collision $1\text{–}4 \rightarrow 2\text{–}3$ is possible and effective but the collision 1–2 has no effect.

The collisions in which N_1 is depleted are:

$$1\text{–}4 \rightarrow 2\text{–}3 \quad 1\text{–}6 \rightarrow 2\text{–}5 \quad 1\text{–}7 \rightarrow 3\text{–}5$$
$$1\text{–}8 \rightarrow 2\text{–}7 \quad 1\text{–}8 \rightarrow 3\text{–}6 \quad 1\text{–}8 \rightarrow 4\text{–}5$$

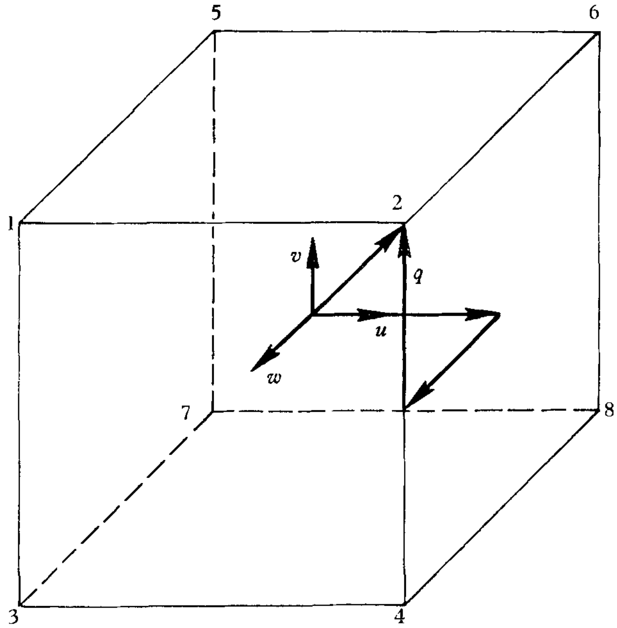

FIGURE 2. The eight-cell model.

Cell ...	1	2	3	4	5	6	7	8
u	−	+	−	+	−	+	−	+
v	+	+	−	−	+	+	−	−
w	+	+	+	+	−	−	−	−

TABLE 1. Sign of velocity components.

and the rate of loss, L_1, is

$$L_1 = v_r^a S_e^a (N_1 N_4 + N_1 N_6 + N_1 N_7) + v_r^b S_e^b (N_1 N_8),$$

where the superscript a refers to the relative velocity and effective cross-section of collision partners diagonally opposed on the faces of the velocity cube in figure 2 and b to those on the cube diagonals.

Assume now that the molecules are hard elastic spheres. Then the scattering from a collision 1–8, for instance, is symmetrical and the molecules are equally distributed to 2–7, 3–6, and 4–5. Therefore the rate of addition to cell 1 is

$$G_1 = v_r^a S_e^a (N_2 N_3 + N_2 N_5 + N_3 N_5) + \tfrac{1}{3} v_r^b S_e^b (N_2 N_7 + N_3 N_6 + N_4 N_5).$$

The relative velocities are given by

$$v_r^a = 2(\tfrac{2}{3})^{\frac{1}{2}}\bar{c}, \quad v_r^b = 2\bar{c}.$$

The effective collision cross-sections are determined as follows. Consider first an encounter of the kind 1–4, which takes place in the $w =$ const. plane, as sketched in figure 1(*b*). The question is what fraction of those molecules that collide should be assigned to cells 2 and 3 (with the others being returned to 1 and 4). Since the scattering has circular symmetry the most reasonable assumption is that half of the colliding molecules are deflected, i.e. that

$$S_e^a = \tfrac{1}{2}S,$$

where S is the mutual collision cross-section. The corresponding assumption of symmetrical scattering for collisions such as 1–8 leads to the result

$$S_e^b = \tfrac{3}{4}S.$$

The flows to be considered are independent of z and are symmetric about the plane $w = 0$; therefore

$$N_1 = N_5, \quad N_2 = N_6, \quad N_3 = N_7, \quad N_4 = N_8.$$

Making use of these conditions and dividing G_1 and L_1 by the number density, n, we get

$$\left(\frac{\partial n_1}{\partial t}\right)_c = \frac{1}{n}(G_1 - L_1) = (1 + (\tfrac{2}{3})^{\frac{1}{2}})\,\bar{c}Sn(n_2 n_3 - n_1 n_4) = 2\theta(n_2 n_3 - n_1 n_4), \qquad (3)$$

where
$$\theta = \tfrac{1}{2}(1 + (\tfrac{2}{3})^{\frac{1}{2}})\,\bar{c}Sn,$$

and n_i is the fraction of molecules in cell i.

It will be convenient to have an expression for θ in terms of $\bar{c}$ and the mean free path in the undisturbed gas, λ. In the undisturbed equilibrium condition all the N's are equal; hence the collision rate of all the molecules of cell 1, for instance, is given by

$$\phi = \bar{c}S[2/3^{\frac{1}{2}}(N_1N_2 + N_1N_3 + N_1N_5) + 2(\tfrac{2}{3})^{\frac{1}{2}}(N_1N_4 + N_1N_6 + N_1N_7) + 2N_1N_8]$$
$$= (6/3^{\frac{1}{2}} + 6(\tfrac{2}{3})^{\frac{1}{2}} + 2)\bar{c}SN_1^2 = 10{\cdot}37\bar{c}SN_1^2.$$

Then the collision frequency per molecule, σ, is

$$\sigma = \phi/N_1 = 10{\cdot}37\bar{c}SN_1 = \frac{10{\cdot}37}{8}\bar{c}Sn,$$

and
$$\lambda = \bar{c}/\sigma = \frac{8}{10{\cdot}37Sn}.$$

Now recall that
$$\theta = \tfrac{1}{2}(1 + (\tfrac{2}{3})^{\frac{1}{2}})\bar{c}Sn,$$

and therefore
$$\theta = \frac{4}{10{\cdot}37}(1 + (\tfrac{2}{3})^{\frac{1}{2}})\bar{c}/\lambda$$
$$= 0{\cdot}70\bar{c}/\lambda.$$

It is clear that θ could be evaluated for other intermolecular force laws but the necessary, somewhat laborious, calculations have not yet been made.

Anticipating the application of equation (2) to low Mach number Couette and Rayleigh flow, let us divide that equation by the number density, n, assumed constant. Then making use of equation (3) and corresponding expressions for the other cells, we get

$$\frac{\partial n_1}{\partial t} - q\frac{\partial n_1}{\partial x} + q\frac{\partial n_1}{\partial y} = 2\theta(n_2 n_3 - n_1 n_4), \tag{4a}$$

$$\frac{\partial n_2}{\partial t} + q\frac{\partial n_2}{\partial x} + q\frac{\partial n_2}{\partial y} = 2\theta(n_1 n_4 - n_2 n_3), \tag{4b}$$

$$\frac{\partial n_3}{\partial t} - q\frac{\partial n_3}{\partial x} - q\frac{\partial n_3}{\partial y} = 2\theta(n_1 n_4 - n_2 n_3), \tag{4c}$$

$$\frac{\partial n_4}{\partial t} + q\frac{\partial n_4}{\partial x} - q\frac{\partial n_4}{\partial y} = 2\theta(n_2 n_3 - n_1 n_4). \tag{4d}$$

Several general features of these equations are worth noting. First, when they are multiplied successively by the collisional invariants, m, mu_i, mv_i (m is the molecular weight) and summed, the right-hand sides vanish, a consequence of the fact that the collisions sketched in figure 1 conserve the number of molecules and satisfy the equations of motion. (Energy is automatically conserved since the speed is constant.)

The equilibrium condition is

$$n_1 n_4 = n_2 n_3,$$

a remnant of the general equilibrium condition

$$f(\mathbf{v}_1')f(\mathbf{v}_2') = f(\mathbf{v}_1)f(\mathbf{v}_2).$$

If the n_i's are independent of x and y it is easy to show with the help of equations (4a)–(4d) that the H function, defined by

$$H = \sum_i n_i \ln n_i,$$

obeys the equation

$$\frac{dH}{dt} = 2\theta(n_2 n_3 - n_1 n_4)(\ln n_1 n_4 - \ln n_2 n_3)$$

and hence

$$\frac{dH}{dt} \leqslant 0$$

as in the exact equation.

3. Couette and Rayleigh flow equations

Now consider the above gas to be contained between two plates moving parallel to the (x, z)-plane or adjacent to one such plate and seek a solution to equations (4a)–(4d) in which the density is constant and the fluid velocity V is zero.

Then, since $V = \sum_i n_i v_i$,

$$n_1 + n_5 + n_2 + n_6 = n_3 + n_7 + n_4 + n_8 = \tfrac{1}{2},$$

and, from the symmetry about $w = 0$,

$$n_1 + n_2 = n_3 + n_4 = \tfrac{1}{4}.$$

These relations make equations ($4b$) and ($4d$), say, superfluous and allow the others to be written

$$\frac{\partial n_1}{\partial t}+q\frac{\partial n_1}{\partial y}=\frac{\theta}{2}(-n_1+n_3), \tag{5a}$$

$$\frac{\partial n_3}{\partial t}-q\frac{\partial n_3}{\partial y}=-\frac{\theta}{2}(-n_1+n_3), \tag{5b}$$

when the terms depending on x are dropped. These are the equations which in the present approximation govern Couette and Rayleigh flows. It is clear that they can approximate the behaviour of a real gas, if at all, only when the plate speed is small relative to $\bar{c}$. In fact, if the wall speed exceeds q the model breaks down completely.

It may be of interest to note that, when the eight-cell discrete velocity approximation is applied to the Krook approximation to the Boltzmann equation (Broadwell 1963), equations identical in form to equations ($5a$) and ($5b$) are found to govern Couette and Rayleigh flows.

4. Couette flow

Consider the steady Couette flow sketched in figure 3. For this problem equations ($5a$) and ($5b$) can be written

$$\frac{dn_1}{dy^*}=\alpha(-n_1+n_3), \tag{6a}$$

$$\frac{dn_3}{dy^*}=\alpha(-n_1+n_3), \tag{6b}$$

where $y^*=y/d$ and $\alpha=\theta d/2q$.

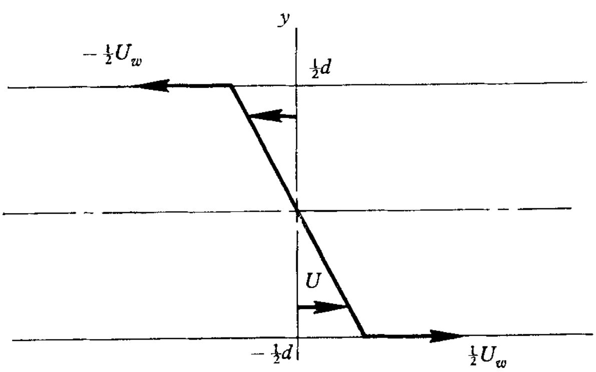

FIGURE 3. Couette flow.

When the reflexion from both walls is diffuse and $n_1=n_5$, $n_2=n_6$, etc., the boundary conditions are

$$q^2[-n_3(\tfrac{1}{2})+n_4(\tfrac{1}{2})]/q[n_3(\tfrac{1}{2})+n_4(\tfrac{1}{2})]=-\tfrac{1}{2}U_w,$$
$$q^2[-n_1(-\tfrac{1}{2})+n_2(-\tfrac{1}{2})]/q[n_1(-\tfrac{1}{2})+n_2(-\tfrac{1}{2})]=\tfrac{1}{2}U_w,$$

where U_w is the difference in the wall velocities. These equations state that the average x-component of velocity of the molecules leaving the upper wall is

$-\frac{1}{2}U_w$ at the upper wall and that of the upward flowing particles is $\frac{1}{2}U_w$ at the lower wall. Then, since $n_1+n_2 = n_3+n_4 = \frac{1}{4}$,

$$n_1(-\tfrac{1}{2}) = \tfrac{1}{8}(1-U_w/2q), \quad n_3(\tfrac{1}{2}) = \tfrac{1}{8}(1+U_w/2q).$$

Solutions to equations (6*a*) and (6*b*) satisfying these conditions are

$$n_1 = \frac{1}{8}\left[\frac{\alpha}{(\alpha+1)}\frac{U_w}{q}y^*+1-\frac{1}{2(\alpha+1)}\frac{U_w}{q}\right],$$

$$n_3 = \frac{1}{8}\left[\frac{\alpha}{(\alpha+1)}\frac{U_w}{q}y^*+1+\frac{1}{2(\alpha+1)}\frac{U_w}{q}\right].$$

From these 'distribution functions' the fluid velocity in the x-direction, U, is found to be

$$U = \sum_i u_i n_i = 2q[-n_1+n_2-n_3+n_4] = q[1-4(n_1+n_3)] = -\frac{\alpha}{(\alpha+1)}U_w y^*, \tag{7}$$

and the shear stress, p_{yx}, is given by

$$p_{yx} = \rho\sum_i u_i v_i n_i = 2\rho q^2[-n_1+n_2+n_3-n_4] = 4\rho q^2[-n_1+n_3]$$
$$= \frac{1}{2(\alpha+1)}\rho q U_w. \tag{8}$$

Defining the viscosity, μ, by $p_{yx}/(dU/dy)$, we have from equations (7) and (8)

$$\mu = \rho q d/2\alpha = \rho q^2/\theta = \tfrac{2}{3}(1+\sqrt{\tfrac{2}{3}})^{-1}\rho\bar{c}/Sn,$$

independent of the degree of rarefaction. With the use of the relation $\theta = 0{\cdot}70\bar{c}/\lambda$, derived above, μ can also be written

$$\mu = 0{\cdot}48\rho\bar{c}\lambda.$$

It seems reasonable, especially after having found this expression for the viscosity, to say that the solutions for the eight-cell model should be compared with those of a real gas having a mean thermal speed equal to $\bar{c}$. This assumption, which is also the usual one in elementary kinetic theory, will be made throughout the paper. Accordingly, the above expression for μ is to be compared with the classical expression:

$$\mu = 0{\cdot}499\rho\bar{c}\lambda.$$

The fractional slip velocity, $2\Delta U/U_w$, defined by

$$2\Delta U/U_w = \frac{\frac{1}{2}U_w - U(-\frac{1}{2})}{\frac{1}{2}U_w}$$

is, from equation (7),

$$2\Delta U/U_w = 1/(\alpha+1) = (0{\cdot}61d/\lambda+1)^{-1}. \tag{9}$$

Lees (1959) found, for Maxwell molecules,

$$2\Delta U/U_w = (0{\cdot}5d/\lambda+1)^{-1}.$$

In this paper Lees made use of the Maxwell integral equation of transfer or moment equation (Maxwell 1890), but departed from the usual procedure of representing the distribution function, f, as a polynomial in the velocity components. Instead, he expressed f in terms of a number of arbitrary functions of space and time which were determined from the moment equation. Lees's expression and equation (9) are shown in figure 4 together with the well-known results of Wang Chang & Uhlenbeck (1954) for Maxwell molecules.

Next, the ratio of the shear stress to the free molecule value, $p_{yx}/(p_{yx})_{\text{f.m.}}$, can be found from equation (8) to be given also by

$$p_{yx}/(p_{yx})_{\text{f.m.}} = 1/(\alpha+1) = (0{\cdot}61 d/\lambda + 1)^{-1}. \tag{10}$$

Similarly, Lees's expression is

$$p_{yx}/(p_{yx})_{\text{f.m.}} = (0{\cdot}5 d/\lambda + 1)^{-1}.$$

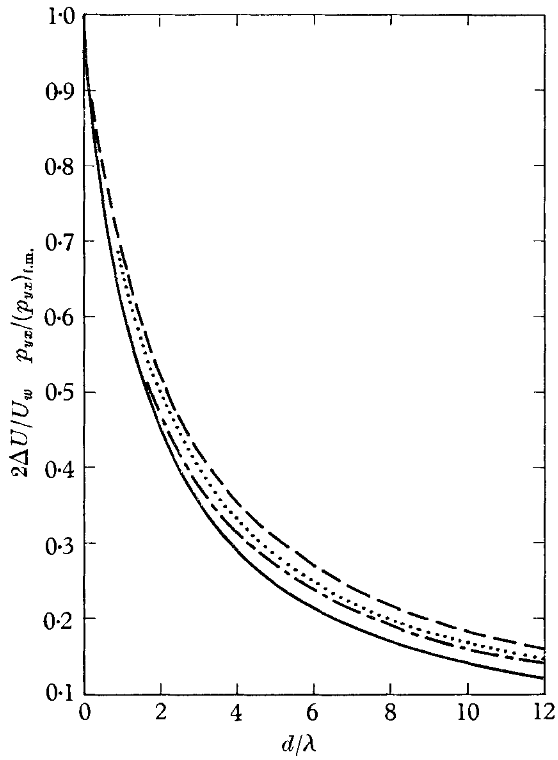

FIGURE 4. Slip velocity and shear stress as a function of Knudsen number in Couette flow. ————, p_{yx}, Wang Chang & Uhlenbeck; ...., p_{yx} and $2\Delta U/U_w$, Lees; ————, $2\Delta U/U_w$, Wang Chang & Uhlenbeck; ——, equations (9) and (10).

In figure 4 the Wang Chang & Uhlenbeck values are compared with these two equations. Equation (10) lies below those of the other investigators at large d/λ because the value for the free molecule shear stress is given incorrectly by equation (8)—it is approximately 15 % too large. The shear stress itself is in good agreement with the other values at large d/λ.

Finally this simple solution allows the clarification of a point not related to low density effects, namely a simple description of the kinetic origin of the stress,

p_{xy} (see figure 5). Almost every text-book dealing with viscous fluid mechanics presents the well-known discussion of the origin of p_{yx}: molecules from below the plane A–B carry a deficit of momentum upward, etc. None, to the author's knowledge, explain p_{xy}.

Consider the molecules crossing plane a–b from right to left. These are molecules with velocities $\mathbf{v}_1$ and $\mathbf{v}_3$. Molecules of class 1 come, on the average, from a region of smaller y and hence smaller U than class 3 molecules. In a region of small U, n_1 and n_3 are relatively large and, of course, in a region of higher U these velocities are relatively depleted. Thus more molecules of class 1 cross a–b than of class 3 and hence positive y-momentum crosses a–b from right to left. The same argument shows that negative y-momentum flows in the opposite direction across a–b and thus an effective positive stress p_{xy} acts on a–b.

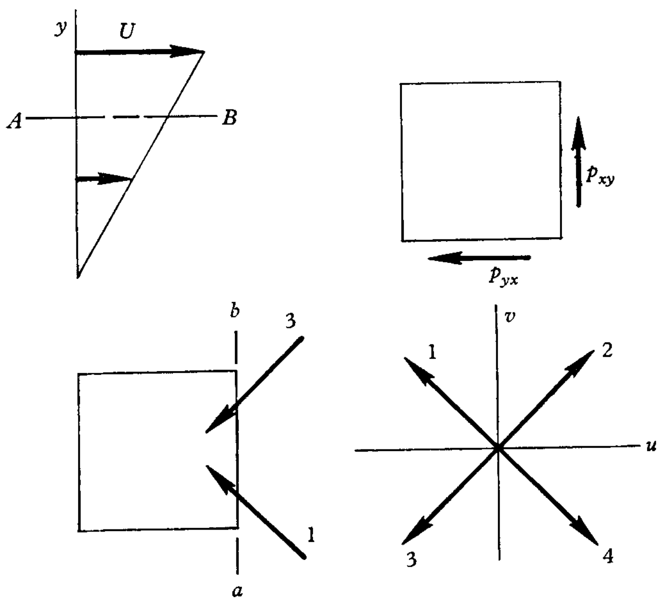

FIGURE 5. The kinetic origin of shear stress.

5. Rayleigh flow

Let the plane $y = 0$ adjacent to a semi-infinite region of gas be suddenly set in motion in its own plane with velocity U_w, assumed small relative to the mean thermal speed $\bar{c}$. Under these conditions the flow is governed by equations (5a) and (5b) rewritten here:

$$\frac{\partial n_1}{\partial t} + q\frac{\partial n_1}{\partial y} = \frac{\theta}{2}(-n_1+n_3), \tag{11a}$$

$$\frac{\partial n_3}{\partial t} - q\frac{\partial n_3}{\partial y} = -\frac{\theta}{2}(-n_1+n_3). \tag{11b}$$

Recall that n_2 and n_4 are determined by

$$n_1+n_2 = n_3+n_4 = \tfrac{1}{4}.$$

The gas is initially at rest so that

$$n_1(y,0) = n_3(y,0) = \tfrac{1}{8}. \tag{12}$$

If the reflexion from the plate is diffuse, then the average x-velocity of the molecules moving upward from the plate is U_w and

$$\frac{q^2[-n_1(0,t)+n_2(0,t)]}{q[n_1(0,t)+n_2(0,t)]} = U_w.$$

Making use of $(n_1+n_2) = \frac{1}{4}$, we find as before

$$n_1(0,t) = \tfrac{1}{8}(1-U_w/q). \tag{13}$$

No condition may be put on n_3 at $y = 0$. From equation (5*a*)

$$n_3 = \frac{2}{\theta}\left(\frac{\partial n_1}{\partial t}+q\frac{\partial n_1}{\partial y}\right)+n_1.$$

Substituting this expression for n_3 in equation (5*b*) we find

$$\frac{1}{q^2}\frac{\partial^2 n_1}{\partial t^2}-\frac{\partial^2 n_1}{\partial y^2}+\frac{\theta}{q^2}\frac{\partial n_1}{\partial t} = 0. \tag{14}$$

Equation (14) is the telegraph equation and is just the equation which Lees (1959) found, by the Maxwell moment method, to govern the fluid velocity, U, and shear stress, p_{yx}, in this problem. In his equation the isothermal speed of sound, $\sqrt{(RT)}$, appears in place of q. Using the equations

$$U = q[1-4(n_1+n_3)], \quad p_{yx} = 4\rho q^2[-n_1+n_3], \qquad (15), (16)$$

it is easy to show that in our case, also, U and p_{yx} are governed by equation (14). This surprising outcome, that a moment method and the present discrete velocity model lead to precisely the same governing equation, is presumably an illustration of the statement by Krook (1955), in connexion with radiation problems, that the two approaches are equivalent.

While, as stated, U and p_{yx} satisfy equation (14), a solution for these quantities cannot be obtained directly because their boundary values are not known.† The boundary and initial conditions on n_1, equations (12) and (13), however, determine a unique solution for this quantity; it may be found in Carslaw & Jaeger (1953).‡ We may anticipate from the form of equation (14) that the solution will have the behaviour which we expect for this model, i.e. that a discontinuity will propagate from the wall at speed q but that for times large compared to the collision time the discontinuity will become extremely small and the solution will have a diffusive character.

Define

$$\tau = \theta t = 0{\cdot}70\bar{c}t/\lambda, \quad \eta = (\theta/q)\,y = 0{\cdot}70\,3^{\frac{1}{2}}y/\lambda.$$

† Lees makes use of a relation between p_{yx} and U on the wall to obtain his results; the complexity of the boundary-value problem is such, however, that only an approximate solution is possible.

‡ The author is grateful to Prof. Lees for pointing out this solution.

In these variables the solution for n_1 is

$$n_1 = \tfrac{1}{8} - \tfrac{1}{8}\,(U_w/q)\,H(\tau-\eta)\left\{e^{-\frac{1}{2}\eta} + \tfrac{1}{2}\eta\int_\eta^\tau \frac{e^{-\frac{1}{2}r}I_1[\frac{1}{2}(r^2-\eta^2)^{\frac{1}{2}}]}{(r^2-\eta^2)^{\frac{1}{2}}}\,dr\right\}, \tag{17}$$

where

$$H(\tau-\eta) = 1 \quad \text{when} \quad \tau > \eta$$
$$= 0 \quad \text{when} \quad \tau \leqslant \eta,$$

and I_1 is the modified Bessel function of first kind of order one. Now n_3 can be found from equation ($11a$) by differentiation; the result is

$$\begin{aligned} n_3 = n_1 - \tfrac{1}{8}\,(U_w/q)\,H(\tau-\eta)&\left\{\frac{\eta\, e^{-\frac{1}{2}\tau}I_1[\frac{1}{2}(\tau^2-\eta^2)^{\frac{1}{2}}]}{(\tau^2-\eta^2)^{\frac{1}{2}}}\right. \\ &- e^{-\frac{1}{2}\eta}(1+\eta/4) + \int_\eta^\tau \frac{e^{-\frac{1}{2}r}I_1[\frac{1}{2}(r^2-\eta^2)^{\frac{1}{2}}]}{(r^2-\eta^2)^{\frac{1}{2}}}\,dr \\ &\left.- \tfrac{1}{2}\eta^2\int_\eta^\tau e^{-\frac{1}{2}r}\left(\frac{I_1'[\frac{1}{2}(r^2-\eta^2)^{\frac{1}{2}}]}{(r^2-\eta^2)} - \frac{2I_1[\frac{1}{2}(r^2-\eta^2)^{\frac{1}{2}}]}{(r^2-\eta^2)^{\frac{3}{2}}}\right)dr\right\}. \end{aligned} \tag{18}$$

Therefore the solution is complete and the fluid velocity and shear stress can be determined from equations (15) and (16). In the following discussion these quantities are compared with those of Yang & Lees (1956), Lees (1959), and Gross & Jackson (1958). In the latter analysis, a moment method was applied in which the distribution function was expressed in half range polynomials.

Notice first that at a given η station n_1 retains its initial value, $\frac{1}{8}$, until the time $\tau = \eta$; then it jumps to the value

$$\tfrac{1}{8}[1-(U_w/q)\,e^{-\frac{1}{2}\tau}].$$

Since n_3 begins to change smoothly at $\tau = \eta$,

$$\frac{U(\tau=\eta)}{U_w} = \tfrac{1}{2}e^{-\frac{1}{2}\tau}, \tag{19}$$

and a discontinuity in U propagates from the wall with velocity q (in t, y co-ordinates) but with a magnitude that decays rapidly beyond a few mean free paths from the wall. The origin of the discontinuity is, of course, in the single molecular speed and it would be instructive to see the influence of more allowed speeds on the initial development of this flow.

Because of the complexity of the general solution it is useful to derive approximate expressions for n_1 and n_3 and thus for U and p_{yx} for short and long times. The asymptotic expansions of the modified Bessel functions in Carslaw & Jaeger (1953) provide the required approximations.

For short times

$$U/U_w \cong \tfrac{1}{2}H(\tau-\eta)\{1-\tfrac{3}{4}\eta+\tfrac{1}{4}\tau+\tfrac{1}{16}\eta^2+\tfrac{1}{8}\eta\tau-\tfrac{1}{16}\tau^2\}, \tag{20}$$

from which it can be seen that the initial gas velocity at the wall is $\frac{1}{2}U_w$, the correct limiting value. In figure 6, velocity profiles given by equation (20) at several fixed times are shown together with one from Lees. As mentioned above, in the latter solution the wave propagation velocity differs slightly from $3^{-\frac{1}{2}}\bar{c}$. Also, the magnitude of the velocity discontinuity on the wave front is given by

$$U/U_w = 0{\cdot}45\, e^{-0{\cdot}557\tau},$$

an expression which again differs little from equation (19). Both these solutions are similar in form to that derived from the Grad thirteen moment equations by Yang & Lees. In particular, in that solution also a rapidly decaying discontinuity in velocity propagates from the wall during the initial period. Quantitatively, however, there are considerable differences; for example, their initial gas velocity at the wall is $0{\cdot}373U_w$.

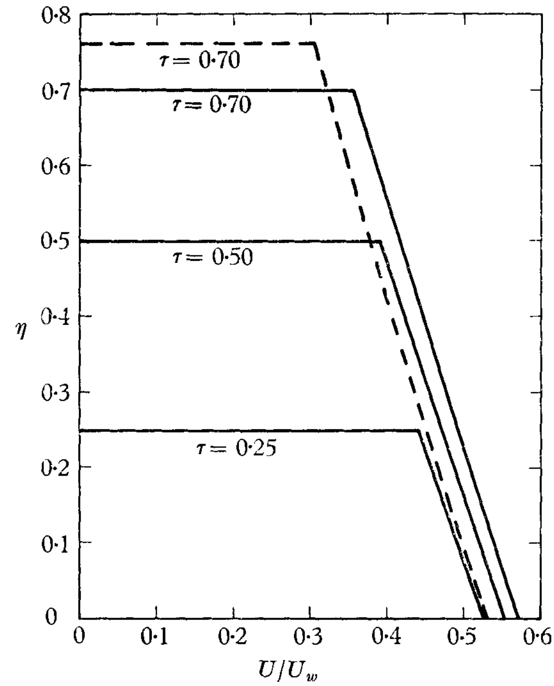

FIGURE 6. Velocity profiles in Rayleigh flow. ----, Lees; ——, equation (20).

In the Gross & Jackson solution, two step functions in flow velocity moving at different speeds appear initially. The gas velocity adjacent to the wall is also $\frac{1}{2}U_w$ initially but rises much more rapidly than the present solution predicts. The existence of the two discontinuities, which would be present in a discrete velocity model with the appropriate velocity cells, is further indication of the similarity between moment and discrete velocity methods.

The limits of the integrals in equations (17) and (18), τ and η, make it more difficult to find the behaviour of n_1 and n_3 for long times. If, however, we allow τ to become large at constant η, we can replace I_1 and I_1' by their asymptotic expansion for large argument and obtain the behaviour of the solution in this limit. To order $\tau^{-\frac{3}{2}}$ the result for the velocity is

$$U/U_w \simeq 1 - \pi^{-\frac{1}{2}}[(1+\eta)\tau^{-\frac{1}{2}} - \tfrac{1}{12}(3+9\eta+3\eta^2+\eta^3)\tau^{-\frac{3}{2}}], \tag{21}$$

an expression which is again similar in form to that found by Yang & Lees. In figure 7, the values of the velocity at $\eta = 0$ given by this equation, and by equation (20) for short times, is compared with the prediction of Lees.

To order $\tau^{-\frac{1}{2}}$ the corresponding expression derived by Gross & Jackson is

$$U/U_w \simeq 1 - \pi^{-\frac{1}{2}}[(1{\cdot}1 + 0{\cdot}98\eta)\tau^{-\frac{1}{2}} - 0{\cdot}26\, e^{-5{\cdot}65\eta\tau^{-\frac{1}{2}}}].$$

Thus the present solution contains part of the correction to the classical Rayleigh solution predicted by Gross & Jackson. It will be interesting to determine whether a discrete velocity model with more cells will yield a solution containing the thin layer adjacent to the wall given by the exponential term in the last equation.

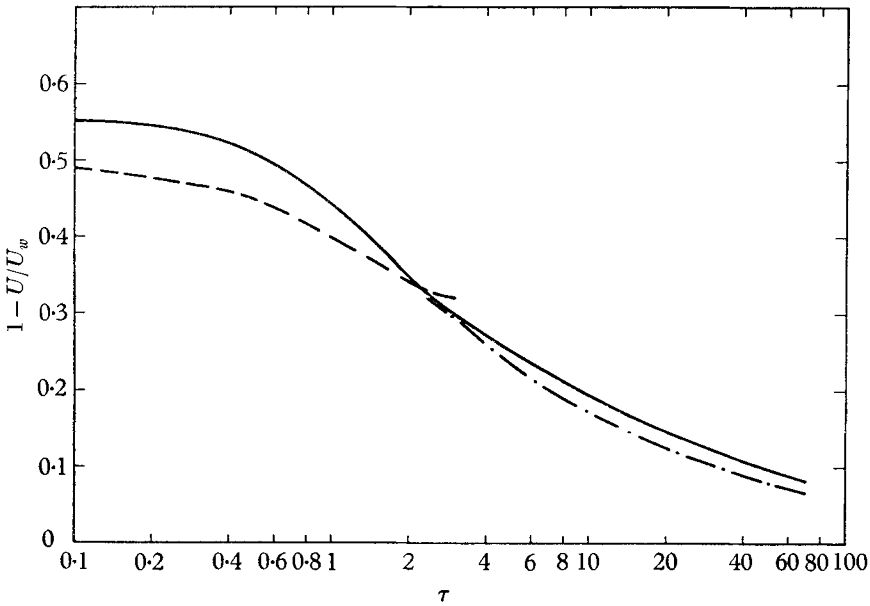

FIGURE 7. The time dependence of the gas velocity at the wall in Rayleigh flow. ——, Lees; ---, equation (20); —·—·, equation (21).

The preceding discussion has dealt only with the flow velocity; in general the shear stress given by equation (16) compares in a similar way to that of the other investigators and an extensive comparison will not be made. Briefly, the skin friction coefficient can be related to the velocity at the wall if n_3 is eliminated from equations (15) and (16). The relation can be put in the form

$$1 - U(0,t)/U_w = C_f U_w/2q = 3^{\frac{1}{2}} C_f U_w/2\bar{c},$$

where $C_f = p_{yx}/\frac{1}{2}\rho U_w^2$. The corresponding expression of Lees is

$$1 - U(0,t)/U_w = C_f U_w/\bar{c}.$$

In conclusion it may be said that the eight-cell discrete velocity model provides a simple physical picture of many of the low density effects in low Mach number shear flow. It is more difficult, in the absence of exact solutions, to evaluate the quantitative results. There seems to be no *a priori* reason, however, to think that the method is less accurate than moment methods, for instance. Furthermore, the model leads to equations that are readily integrated by machine. Preliminary machine solutions describing the initiation and internal structure of shock waves and compressible Rayleigh flow have been obtained for models with up to forty molecular velocities. The study of these flows is continuing, as are attempts to clarify the relation between moment and discrete velocity methods.

The author is pleased to acknowledge the help of Mr Frank Vogenitz in the mathematical aspects of the analysis.

REFERENCES

Broadwell, J. E. 1963 Space Technology Laboratories Report No. 9813-6001-RU000.

Carslaw, H. S. & Jaeger, J. C. 1953 *Operational Methods in Applied Mathematics*. Oxford University Press.

Chandrasekhar, S. 1960. *Radiative Transfer*. New York: Dover.

Gross, E. P. & Jackson, E. A. 1958 *Phys. Fluids*, **1**, 318.

Gross, E. P. 1960 *Rarefied Gas Dynamics*, Vol. 3, ed. F. M. Devienne. New York: Pergamon Press.

Krook, M. 1955 *Astrophys. J.* **122**, 488.

Lees, L. 1959 California Institute of Technology Hypersonic Research Project Memo. no. 51.

Maxwell, J. C. 1890 *Scientific Papers*, vol. II, p. 26. Cambridge University Press.

Wang Chang, C. S. & Uhlenbeck, G. E. 1954 University of Michigan Engr. Res. Inst. Report No. 1999-1-T.

Yang, H. & Lees, L. 1956 *J. Math. Phys.* **35**, 195.